Numerical Solutions of Freholm Integral Equations of the First Kind by Using Coiflets

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Abstract

Many articles have addressed different methods to solve Freholm integral equations, but none have used coiflets and interpolation methods. In this article, we use a scaling function interpolation method to solve linear Fredholm integral equations of the first kind, and we prove a convergence theorem for the solution of Fredholm integral equations. We present three examples which have better results than previous article.

Keywords

Freholm Integral Equation; Coiflets; Wavelets; Scaling Function Interpolation; Error Estimates

Introduction

Recently, the study of the Fredholm integral equations has attracted the attention of mathematicians and researchers in other disciplines such as [6, 7], where many problems lead to the solution of integral equations [1-3]. Integral equations are also useful in many branches of pure mathematics.

Wavelets have been applied in a wide range of engineering and physical disciplines, and are a useful tool for scientists and mathematicians. In this paper we will find the numerical solution for linear Fredholm of the first kind of the form

$$f(x) = \int_a^b k(x,t) y(t) dt, \qquad (1)$$

where the functions $f(x)$ and $k(x,t)$ are given functions over the given interval $[a,b]$ and the square respectively, such that the function $k(x,t)$ is called the kernel and the unknown function $y(t)$ is to be determined.

Coiflet

In this section we review briefly on wavelet transform and Multiresolution Analysis (MRA) [4]. We first define the scaling function $\phi(x)$ and the sequence $\{\alpha_p, p \in Z\}$ such that

$$\phi(x) = \sum_p \alpha_p \phi(2^j x - p) = \sum_p \alpha_p \phi_{j,p}(x) \qquad (2)$$

By using this dilation and translation [4], we defined a nested sequence spaces $\{V_j, j \in Z\}$ which is called MRA of $L^2(R)$ with the following properties

$$V_j \subset V_{j+1}, j \in Z \qquad (3)$$

$$V_{-\infty} = \bigcap_{j \in Z} V_j = \{0\} \qquad (4)$$

$$\bigcup_j V_j \text{ is dense in } L^2(R) \qquad (5)$$

$$\phi(x) \in V_j \Leftrightarrow \phi(2x) \in V_{j+1} \qquad (6)$$

For the subspace V_1 is built by $\phi(2x - p)$, $p \in Z$ then $V_0 = \{\phi(x - p), p \in Z\}$ And; since $V_0 \subset V_1$, we have

$$\phi(x) = \sum_p \alpha_p \phi(2x - p)$$

.

In general,

$$\phi(x) = \sum_p \alpha_p \phi(2^j x - p) = \sum_p \alpha_p \phi_{j,p}(x) \qquad (7)$$

which shows (2) is well defined. Any function $f(x) \in L^2(R)$ can be approximated by scaling functions in one of the subspaces in the given nested sequence. In fact, for each j we define the orthogonal complement subspace W_j of V_j in the subspace V_{j+1}. The orthogonal basis of W_j is denoted by

$$\psi_{j,p}(x) = \psi(2^j x - p), \qquad (8)$$

and the wavelet function can be obtained by

$$\psi(x) = \sum_p \beta_p \phi_{j,p}(x). \qquad (9)$$

More precisely, V_j is the subspace generated by $\{\phi_j\}$.

Some interesting properties of scaling and wavelet functions make wavelet method more efficient than quadrature formula methods, spline approximations or other methods in solving Integral equations. A lot of computational time and storage capacity can be saved since we do not require a huge number of arithmetic operations partly due to the following properties.

(i) Vanishing Moments:

$$\int_{-\infty}^{\infty} x^k \psi(x) dx = 0, \; k = 0, \ldots, m-1. \qquad (10)$$

and in this case the wavelet is said to have a vanishing moment of order m,

(ii) Semiorthogonality:

$$\langle \psi_{i,p}(x), \psi_{j,k}(x) \rangle = \int_{-\infty}^{\infty} \psi_{j,p}(x) \psi_{j,k}(x) dx = 0; \qquad (11)$$

$$p \neq k; \; j, k, p \in Z.$$

The set of scaling functions $\{\phi_{n,\cdot}\}$ are orthogonal at the same level n, which means:

$$\langle \phi_{n,k}(x), \phi_{n,p}(x) \rangle = \int_{-\infty}^{\infty} \phi_{n,k}(x), \phi_{n,p}(x) dx = \delta_{k,p}, \qquad (12)$$

$$n, k, p \in Z$$

Coiflet (of order L) has more symmetries and is an orthogonal multiresolution wavelet system with,

$$M_k = \int x^k \phi(x) dx = 0, \; k = 1, 2, \ldots, L-1. \qquad (13)$$

$$\int x^k \psi(x) dx = 0, \quad k = 0, 1, \ldots L-1. \qquad (14)$$

Where $\{M_k\}$ are the moments of the scaling functions.

Scaling Function Interpolation

The function $f(x)$ can be interpolated by using the basis functions in the subspace V_j as follows.

$$f^j(x) = \sum_p a_p \phi(2^j x - p) \qquad (15)$$

where a_p are the coefficients evaluated by using equation (12) such that

$$a_p = \langle f(x), \phi_{j,p}(x) \rangle = \int f(x) \phi(2^j x - p) dx. \qquad (16)$$

Hence the equation (15) becomes:

$$f^j(x) = \sum_p \left(\int f(x) \phi(2^j x - p) dx \right) \phi(2^j x - p).$$

On the other hand, one can use sampling values of f at certain points to approximate the function f. It has been proved in [5], namely, an interpolation theorem using coiflet such that if $\phi(x)$ and $\psi(x)$ are sufficiently smooth and satisfy the equations (10)-(14) and the function $f(x) \in C^k(\overline{\Omega})$, where Ω is a bounded open set in R^2, $k \geq N \geq 2$, $j \in Z$ Then,

$$f^j(x, y) = \frac{1}{2} \sum_{p, q \in \Lambda} f\left(\frac{p+c}{2^j}, \frac{q+c}{2^j} \right) \phi_{j,p}(x) \phi_{j,q}(y),$$

$$(x, y) \in \Omega \qquad (17)$$

where the index set is

$$\Lambda = \{ (p, q) | (\text{supp}(\phi_{j,p}) \otimes \text{supp}(\phi_{j,q})) \cap \Omega \neq \phi \}.$$

In addition, the moment M_l satisfies

$$M_l = (c)^l, \; l = 1, 2, \ldots, N-1,$$

$$c = M_1.$$

Then,

$$\| f - f^j \|_{L^2(\Omega)} \leq C \| f^{(N)} \|_\infty \left(\frac{1}{2^j} \right)^N. \qquad (18)$$

Where C is a constant depending only on N, diameter of Ω and

$$\left\| f^{(N)} \right\|_\infty := \max_{(x,y) \in \Omega, m=0, \ldots, N} \left| \frac{\partial^N f}{\partial x^m \partial y^{N-m}} (x, y) \right|. \qquad (19)$$

For the function with one variable, using coiflet (c=0), we have

$$f^j(x) = \frac{1}{2^j} \sum_p f\left(\frac{p}{2^j} \right) \phi_{j,p}(x), \; x \in [a, b], \qquad (20)$$

and

$$\left| f - f^j \right|_{L^2[a,b]} \leq C \left\| f^{(N)} \right\|_\infty \left(\frac{1}{2} \right)^N. \qquad (21)$$

Where $\left\|f^{(N)}\right\|_\infty := \max\limits_{x\in(a,b),m=0,\ldots,N}\left|\dfrac{\partial^N f}{\partial x^m}(x)\right|.$

Solutions of Fredholm Integral Equation of the First Kind Using Coiflet

In this section we will apply coiflet and the interpolation formula (20) to solve the Fredholm integral equation (1). The unknown function $y(x)$ in equation (1) can be expressed in term of the scaling functions $\phi_{j,p}(x)$ in the subspace V_j such that

$$y^j(x) = \sum_p a_p \phi_{j,p}(x). \qquad (22)$$

Consider the equation (1) and the function $y(x)$ which is defined on the interval [a, b] and the scaling function $\phi(x)$ is supported with (d_1, d_2), then we have the index:

$\Lambda = \{2^j b - d_2, 2^j b - d_2 + 1, \ldots\ldots, 2^j a - d_1\}$. This coincides with the two dimension index set shown above. The reason is as follows. Since the support of ϕ is (d_1, d_2), the nonzero part of $\phi_{j,p}(x)$ is located on the interval which satisfies $d_1 < 2^j x - p < d_2$. Hence $\dfrac{d_1 + p}{2^j} < x < \dfrac{d_2 + p}{2^j}.$ On the other hand, the unknown function is defined on the interval [a, b]. This implies $\dfrac{d_1 + p}{2^j} < a$ and $b < \dfrac{d_2 + p}{2^j}.$

Consequently, $p < 2^j a - d_1$ and $2^j b - d_2 < p$. This proves the index set is shown as above.

By applying equation (22) into equation (1), we get the system, $f(x) = \int_a^b k(x,t)\sum_p a_p \phi(2^j t - p)dt, \qquad (23)$

$$f(x) = \sum_{p\in\Lambda} a_p \int_a^b k(x,t)\phi(2^j t - p)dt), \qquad (24)$$

where the coefficients $\{a_p, p \in \Lambda\}$ can be evaluated by substituting $\{x_p \in [a,b], p \in \Lambda\}$ into the system.

This gives rise to coefficients in (24) and we obtain a numerical solution of (1). The error analysis of this method is obtained in the following section.

Error Analysis

In this section will discuss the convergence rate of our method for solving linear Fredholm integral equation of the first kind (1) by using coiflets.

Theorem1. In equation (1), suppose that the function $k(x,t)\in C([a,b]\times[a,b])$, is positive for all $x,t \in [a,b]$, Such that

$$m_1 = \min\{k(x,t), (x,t) \in [a,b]\text{x}[a,b]\}.$$

Given $\quad f(x) \in C[a,b], for\ j \in Z,$

$$y^j(x) = \sum_p a_p \phi(2^j x - p) \qquad (25)$$

is an approximate solution of the equation (1) with the coefficients obtained in (24). Then,

$$\left\|e(x)\right\| = \left\|y(x) - y^j(x)\right\| \le C(\frac{1}{2})^j \qquad (26)$$

where,

$$\| e(x) \| = \int_a^b | e(x) |\, dx$$

Proof. Subtracting equation (25) from equation (1) and taking the norm for both sides, we get the following

$$e(t) = \sum_p a_p \phi_{i,p}(t) - y(t)$$

$$K = \int_a^b k(x,t)e(t)dt = \int_a^b k(x,t)(\sum_p a_p \varphi_{i,p}(t) - y(t))dt$$

$$\| K \| = \| \int_a^b k(x,t)(\sum_p a_p \varphi_{i,p}(t) - y(t))dt \| \text{ (27)}.$$

But $k(x,t) > m_1$ then, $\int_a^b k(x,t)e(t)dt > m_1\int_a^b e(t)dt,$

This leads to

$$\left\|\int_a^b k(x,t)e(t)dt\right\| > m_1\left\|e\right\|$$

Then equation (27) becomes:

$$\left\|e\right\| \le \frac{1}{m_1}\left\|\int_a^b k(x,t)dt\right\|\left\|\int_a^b (\sum_p a_p \phi(2^j t - p) - y(t))dt\right\|$$

$$= c_1 \left\| \int_a^b (\sum_p a_p \phi(2^j t - p) - y(t)) dt \right\| \qquad (28)$$

where $c_1 = \dfrac{1}{m_1} \left\| \int_a^b k(x,t) dt \right\|$

By [5], the unknown function $y(x)$ can be interpolated by using coiflet such that:

$$y^j(x) = \sum_p y(\frac{p}{2^j}) \phi(2^j x - p). \qquad (29)$$

Let $t = x$ in equation (28) then adding and subtracting the quantity $\sum_p y(\frac{p}{2^j}) \phi(2^j t - p)$ and using the triangle inequality in equation (27), we get the following inequality.

$$\|e(x)\| \leq c_1 \left\| \int_a^b (\sum_p a_p \phi(2^j t - p) - y(t) \right.$$

$$\left. + \sum_p y(\frac{p}{2^j}) \phi(2^j t - p) - \sum_p y(\frac{p}{2^j}) \phi(2^j t - p)) dt \right\|$$

$$\leq c_1 \left(\left\| \int_a^b (\sum_p y(\frac{p}{2^j}) \phi(2^j t - p) - y(t)) dt \right\| \right.$$

$$\left. + \left\| \int_a^b (\sum_p y(\frac{p}{2^j}) \phi(2^j t - p) - \sum_p a_p \phi(2^j t - p) dt \right\| \right) \qquad (30)$$

which equals to the following

$$c_1 \left(\left\| \int_a^b y(\frac{p}{2^j}) \phi(2^j t - p) - y(t)) dt \right\| \right.$$

$$\left. + \left\| \sum_p (y(\frac{p}{2^j}) - a_p) \int_a^b \phi(2^j t - p) dt \right\| \right).$$

By Theorem 3.1 of [5] and using coiflets, we have

$$\left\| \sum_p y(\frac{p}{2^j}) \phi(2^j t - p) - y(t) \right\| \leq c_0 \|f^{(N)}\|_\infty (\frac{1}{2^j})^N. \qquad (31)$$

Since $\sum_p (y(\frac{p}{2^j}) - a_p)$ is finite we define it as

$$\sum_p (y(\frac{p}{2^j}) - a_p) = c_2. \qquad (32)$$

Using the above results and the orthonomality of the scaling functions $\{\phi_{j,p}(x)\}$ we conclude that

$$\|e(x)\| \leq c_1 (c_0 \|f^{(N)}\| (\frac{1}{2^j})^N + c_2 (\frac{1}{2})^j) = c(\frac{1}{2})^j.$$

Where c is some constant independent of j.

Numerical Examples

In the following examples, we will solve linear Fredholm integral equation (1) using coiflet of order 5 and provide errors between exact solutions and numerical solutions at different resolution levels. These examples are also presented in [7] by using different methods. We will compare our results with the results in [7].

Example 1

Consider the integral equation (1) with $f(x) = e^x + (1 - e)x - 1$,

$$k(x,t) = \begin{cases} t(x-1) & t < x \\ x(t-1) & x \leq t \end{cases}$$

and the exact solution $y(x) = e^x$. The numerical results are shown in Table 1.

Example 2

Consider the integral equation

$$\frac{(1+x^2)^{\frac{3}{2}} - x^3}{3} = \int_0^1 (x^2 + t^2)^{\frac{1}{2}} y(t) dt,$$

where the exact solution is $y(x) = x$ and the numerical results are shown in Table 2.

Example 3

In the following integral equation

$$\frac{e^{x+1} - 1}{x+1} = \int_0^1 e^{xt} y(t) dt$$

with the exact solution $y(x) = e^x$. We present the numerical results in Table 3.

In what follows, we present our numerical solutions at different levels of resolutions for the above three examples.

TABLE 1 NUMERICAL RESULTS FOR EXAMPLE 1

x_i	Exact solution	Approximation Solution		
		j=-2	j=-1	j=0
0.0	1	1.00001	1	1
0.1	1.105171	1.10516	1.10517	1.10517
0.2	1.221403	1.22139	1.2214	1.2214
0.3	1.349859	1.34986	1.34986	1.34986
0.4	1.491825	1.49189	1.49183	1.49183
0.5	1.648721	1.64873	1.64872	1.64872
0.6	1.822119	1.82211	1.82212	1.82212
0.7	2.013753	2.01377	2.01375	2.01375
0.8	2.225541	2.22556	2.22554	2.22554
0.9	2.459603	2.45964	2.45961	2.4596
1	2.718282	2.7183	2.71829	2.71828

TABLE 2 NUMERICAL RESULTS FOR EXAMPLE 2

x_i	Exact Solution	Approximation Solution		
		j=-2	j=-1	j=0
0.0	0.0	0.0	0.0	0.0
0.1	0.1	0.1	0.1	0.1
0.2	0.2	0.2	0.2	0.2
0.3	0.3	0.3	0.3	0.3
0.4	0.4	0.4	0.4	0.4
0.5	0.5	0.5	0.5	0.5
0.6	0.6	0.6	0.6	0.6
0.7	0.7	0.7	0.7	0.7
0.8	0.8	0.8	0.8	0.8
0.9	0.9	0.9	0.9	0.9
1	1	1	1	1

TABLE 3 NUMERICAL RESULTS FOR EXAMPLE 3

x_i	Exact Solution	Approximation Solution		
		j=-2	j=-1	j=0
0.0	1	1.00001	1.00009	0.99997
0.1	1.105171	1.10516	1.10517	1.10517
0.2	1.221403	1.22139	1.2214	1.2214
0.3	1.349859	1.34986	1.34986	1.34986
0.4	1.491825	1.49189	1.49183	1.49183
0.5	1.648721	1.64873	1.64872	1.64872
0.6	1.822119	1.82211	1.82212	1.82212
0.7	2.013753	2.01377	2.01375	2.01375
0.8	2.225541	2.22556	2.22554	2.22554
0.9	2.459603	2.45964	2.45961	2.4596
1	2.718282	2.7183	2.71829	2.71828

In the following two tables we will compare our results with [7] for the first and third examples. The results in the second example are close to the perfect solution.

TABLE 4 COMPARISONS OF EXAMPLE 1 WITH OTHER METHOD

The absolute error for example 1				
	Our method		Method in [7]	
x_i	j=-2	j=-1	j=0	
0.0	1.0E-5	0	0	4.07E-2	6E-4
0.1	1.1E-5	1.0E-6	1.0E-6	9.99E-2	1.0E-6
0.2	1.3E-5	3.0E-6	3.0E-6	7.7E-5	3.0E-6
0.3	1.0E-6	1.0E-6	1.0E-6	1.41E-1	1.0E-6
0.4	6.5E-5	5.0E-6	5.0E-6	2.5E-5	5.0E-6
0.5	9.0E-6	1.0E-6	1.0E-6	7.71E-4	1.0E-6
0.6	9.0E-6	1.0E-6	1.0E-6	8.4E-4	1.0E-6
0.7	1.7E-5	3.0E-6	3.0E-6	1.52E-3	3.0E-6
0.8	1.9E-5	1.0E-6	1.0E-6	8.71E-4	1.0E-6
0.9	3.7E-5	7.0E-6	3.0E-6	1.60E-2	7.0E-6

Where E-n denotes 10^{-n}.

TABLE 5 COMPARISONS OF EXAMPLE 3 WITH OTHER METHOD

The absolute error for example 3				
	Our method		Method in [7]	
x_i	j=-2	j=-1	j=0	
0.0	1.0E-5	9.0E-5	3.0E-5	1.11E-3	5E-5
0.1	1.1E-5	1.0E-6	1.0E-6	5.0E-4	4.1E-5
0.2	1.3E-5	3.0E-6	3.0E-6	3.33E-4	5.7E-5
0.3	1.0E-6	1.0E-6	1.0E-6	4.6E-4	6.9E-5
0.4	6.5E-5	5.0E-6	5.0E-6	6.0E-4	5.5E-4
0.5	9.0E-6	1.0E-6	1.0E-6	2.25E-3	2.4E-4
0.6	9.0E-6	1.0E-6	1.0E-6	8.39E-4	6.9E-5
0.7	1.7E-5	3.0E-6	3.0E-6	7.83E-4	8.7E-5
0.8	1.9E-5	1.0E-6	1.0E-6	4.9E-4	8.8E-4
0.9	3.7E-5	7.0E-6	3.0E-6	8.6E-4	8.7E-5

Conclusion

In this work, we applied scaling function interpolation method by using coiflets to solve Fredholm integral equations of the first kind. As it is shown in Table 4 and Table 5, we obtained better results than [7]. Our method can be applied to different kinds of integral equations, integral-algebraic equations and integral equations of two dimensions which will be an efficient method for image processing. In addition, we can interpolate the given functions in the integral equation. This would simplify the calculations in finding numerical solutions of integral equations. Using our method to solve nonlinear integral equations would be interesting as well.

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REFERENCES

[1] F. Brauer, "On a nonlinear Integral Equation for Population Growth Problems," SIAM, N.6, 2000, pp. 312-317.

[2] F. Brauer and C. Castillo, "Mathematical Models in population biology and epidemiology," Applied Mathematics and Computation, Springer-Verlang., New York, 2001.

[3] T. A. Butorn, "Volterra Integral and differential equations," Academic Press., New York, 1983.

[4] C. K. Chui, "An Introduction to Wavelets," Academic Press, 1992.

[5] E, B. Lin and X. Zhou, "Coiflet Interpolation and Approximate Solutions of Elliptic Partial Differential Equations," Numerical Methods for Partial Differential Equations,Vol .13, No.4, 1997, pp.302-320.

[6] K. Maleknjaf and T. Lotfi, "Using Wavelet For Numerical Solution of Fredholm Integral Equations," Proceedings of the World Congress on Engineering, Vol. 2, London, , U.K, 2-4 July 2007,pp.2-6.

[7] K. Maleknjad and S. Sohrabi, "Numerical Solution of Fredholm Integral Equations of the First Kind by Using Legender Wavelet," Applied Mathematics and Computation, 186, 2007, pp836-843.